Vector Spaces and Subspaces

$\mathbb{R}^n$ is called a vector space.

A set containing objects (vectors) along with notions of addition and scalar multiplication which satisfy the usual algebraic properties (commutativity, associativity, distribution, etc)

There are other vector spaces, but in this class, we will focus on $\mathbb{R}^n$.

(come to office hours if you'd like to learn more!)

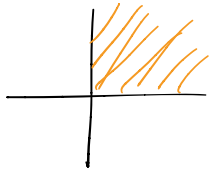
Def: A subspace W of $\mathbb{R}^n$ is a subset such that

1) $0 \in W$ (W is nonempty)
2) If $u, v \in W$, then $u+v \in W$ (W is closed under addition)
3) If $v \in W$, $c \in \mathbb{R}$, then $cv \in W$ (W is closed under scalar mult.)

Ex:

- $\{0\}$ is a subspace of $\mathbb{R}^n$ (trivial/zero subspace)
  since $0 \in \{0\}$, $0+0 \in \{0\}$, and $c \cdot 0 = 0 \in \{0\}$ $\forall c \in \mathbb{R}$.
- $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ (check)

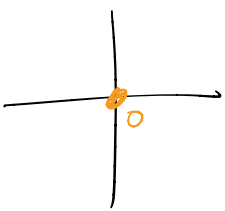
Ex: Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x, y \geq 0 \right\}$. ← "such that"



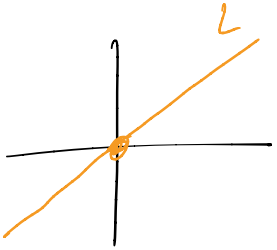
W is not a subspace of $\mathbb{R}^2$

because $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in W$ but $-1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin W$.

Subspaces of $\mathbb{R}^2$:



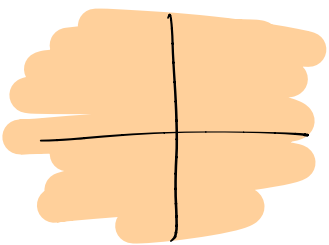
$\{0\}$



Lines through origin:

- $0 \in L$ ✓
- $v, w \in L \Rightarrow v + w \in L$
- $v \in L,\ c \in \mathbb{R} \Rightarrow cv \in L$

} Can see geometrically



$\mathbb{R}^2$ itself

Def: Let $v_1, \dots, v_p \in \mathbb{R}^n$. The span of $v_1, \dots, v_p$ is the set of all linear combinations of $v_1, \dots, v_p$.

$$\text{Span}\{v_1, \dots, v_p\} = \left\{ x_1 v_1 + \cdots + x_p v_p \;\middle|\; x_1, \dots, x_n \in \mathbb{R} \right\}$$

Ex:

- Span $\{0\} = \{0\}$ = origin
- Let $v \neq 0$. Then

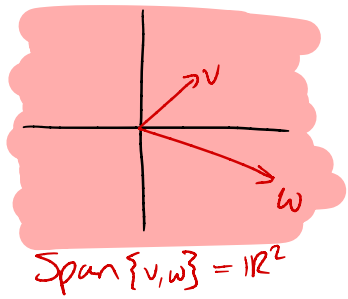
  Span $\{v\} = \{xv \mid x \in \mathbb{R}\}$

  = line through origin

- Let $v, w \neq 0$, v not a scalar multiple of w. Then

$$\text{Span}\{v, w\} = \{x_1 v + x_2 w \mid x_1, x_2 \in \mathbb{R}\}$$

$$= \text{plane through origin}$$



Key Point: If $v_1, \ldots, v_p \in \mathbb{R}^n$, then $\text{Span}\{v_1, \ldots, v_p\}$ is a subspace of $\mathbb{R}^n$.

proof:

- $0 = 0v_1 + 0v_2 + \cdots + 0v_p \in \text{Span}\{v_1, \ldots, v_p\}$
- Let $u = x_1 v_1 + \cdots + x_p v_p$, $w = y_1 v_1 + \cdots + y_p v_p \in \text{Span}\{v_1, \ldots, v_p\}$

  then $u + w = (x_1 v_1 + \cdots + x_p v_p) + (y_1 v_1 + \cdots + y_p v_p)$

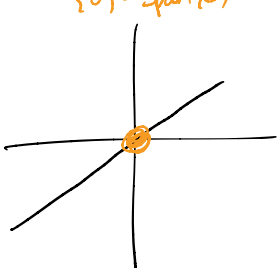
  $= (x_1 + y_1) v_1 + \cdots + (x_p + y_p) v_p \in \text{Span}\{v_1, \ldots, v_p\}$
- Let $u = x_1 v_1 + \cdots + x_p v_p \in \text{Span}\{v_1, \ldots, v_p\}$, $c \in \mathbb{R}$

  then $cu = c(x_1 v_1 + \cdots + x_p v_p)$

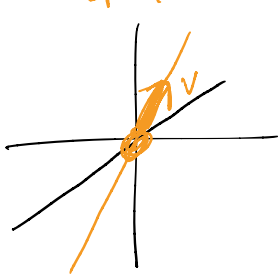
  $= (cx_1) v_1 + \cdots + (cx_p) v_p \in \text{Span}\{v_1, \ldots, v_p\}$ ∎

Spans in $\mathbb{R}^3$

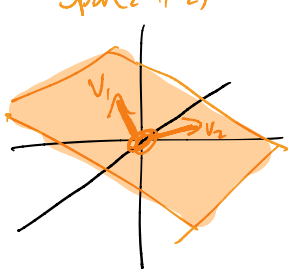
$\{0\} = \text{Span}\{0\}$



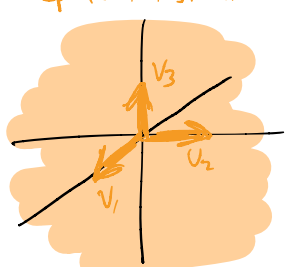
$\text{Span}\{v\}$



$\text{Span}\{v_1, v_2\}$



$\text{Span}\{v_1, v_2, v_3\} = \mathbb{R}^3$



Important Question: How do we determine whether a given vector b is in the span of some vectors?

$$b \in \text{Span}\{v_1, \dots, v_p\}$$
$$\iff b = x_1 v_1 + \cdots + x_p v_p \quad \text{for some } x_1, \dots, x_p \in \mathbb{R}$$
$$\iff b = \underbrace{[v_1 \ \cdots \ v_p]}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}}_{x}$$
$$\iff b = Ax \quad \text{for some } x \in \mathbb{R}^p$$

$b \in \text{Span}\{v_1, \dots, v_p\} \iff Ax = b$ has a solution where $A = [v_1 \ \cdots \ v_p]$

Ex: Is $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \in \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right\}$?

(ie. is $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ a L.C. of $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$?)

Solve $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & 2 \\ -1 & 0 & -2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 2 \\ x_2 = 1 \end{array}$$

Since there is a solution, $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ is in the span. Moreover, the solution allows us to express $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ as a L.C.: $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$